On nonlinear Schrödinger equations with random potentials: existence and probabilistic properties

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Abstract

In this paper we are concerned with nonlinear Schrödinger equations with random potentials. Our class includes continuum and discrete potentials. Conditions on the potential V_ω are found for existence of solutions almost sure ω. We study probabilistic properties like central limit theorem and law of larger numbers for the obtained solutions by independent ensembles. We also give estimates on the expected value for the L^∞-norm of the solution showing how it depends on the size of the potential.

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1 Introduction

A class of models that appears naturally in a wide number of phenomena are the random differential equations. This occurs because randomness is a powerful tool and concept to control complex systems involving a large number of variables and particles. The basic idea is describe complex systems by means of their statistical properties. Another kind of phenomena are those governed by quantum mechanics and uncertainty principle. In this direction, we have Schrödinger equations, and their random versions, which are core in the study of condensed matter.

In this paper we are concerned with a random version of the nonlinear Schrödinger equation

$$ih\frac{\partial \psi}{\partial t} = -h^2\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $t \in \mathbb{R}$, $n \geq 3$, $p > 1$, h is the Planck constant and i is the imaginary unit. When looking for standing wave solutions, namely those which have the special form $\psi(x,t) := e^{-i\frac{E}{h}t}u(x)$, with $E \in \mathbb{R}$, we are leading to solve the following stationary equation

$$-\Delta u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N.$$

From the physical viewpoint, the function V is the potential energy, and therefore the force acting on the system is given by $F(x) = -\nabla V(x)$. In the deterministic case, there are many papers concerning existence, multiplicity and qualitative properties for the solution of the above equation (see [19, 11, 2, 1] and references therein).

The main interest of this paper is to study situations where the potential V is not deterministic. Worth to mention that during the last thirty years, random Schrödinger operators, which originated in condensed matter physics, have been studied intensively by physicists and mathematicians. The theory is at the crossroads of a number of mathematical fields: the theory of operators, partial differential equations, the theory of probabilities and also stochastic process. This paper aims to prove the existence and probabilistic properties of bounded solutions for the random equation

$$\begin{cases} -\Delta u + V_\omega(x)u &= b(x)u|u|^{p-1} + g(x), \quad \text{if } x \in U; \\ u &= 0, \quad \text{if } x \in \partial U, \end{cases} \tag{1.2}$$

where V_ω is a random variable, $U \subset \mathbb{R}^n$ is a bounded domain and the terms b, $g \in L^\infty(U)$ are deterministic. In fact, the boundedness of U is not essential and could be circumvented by working in weighted L^∞-spaces or Lebesgue spaces $L^s(\mathbb{R}^n)$ with $s \neq \infty$ (see [13, 14]). However, here this condition will simplify matters a bit. The random potential V_ω is constructed via a convolution with a realization of a random variable valued in the finite random measure space. Precisely, given a continuous function $f : \mathbb{R}^N \to \mathbb{R}$ we consider

$$V_\omega(x) := \int_U f(x-y)\, d\mu_\omega(y) \tag{1.3}$$

where μ_ω is a $\mathcal{M}(U)$-valued random variable.

We present here some examples of (1.3) that have been treated in the literature (see e.g. the review [17]). We first consider a model of an unordered alloy, that is, a mixture of several materials with atoms located at lattice positions. If we assume that the type of atom at the lattice $i \in \mathbb{Z}^n$ is random we are leading to consider the following type of potential

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^n} q_i(\omega) f(x-i), \tag{1.4}$$

where the random variables q_i describe the charge of the atom at the position i of the lattice. Other example can be obtained if we consider materials like glass or rubber, where the position of the atoms of the material are located at random points η_i in space. By normalizing the charge of the atoms, the suggested potential is formally

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^n} f(x - \eta_i(\omega)), \tag{1.5}$$

where the $\eta_i(\omega)$ are random variables which localize the atoms in the spaces.

The class of potentials allowed here is sufficient large to consider many known models. For example, the case of glass considered in (1.5) can be obtained if we take the random point measure $\mu_\omega = \sum_i \delta_{\eta_i(\omega)}$. Actually, for this choice of the measure we have that

$$\sum_{i\in\mathbb{Z}^n\cap U} f(x-\eta_i(\omega)) = \int_U f(x-\eta)d\mu_\omega(\eta). \tag{1.6}$$

Also, a combination of potentials like (1.4) and (1.5), namely $\Sigma_{i\in\mathbb{Z}^n\cap U} q_i(\omega)f(x-\eta_i(\omega))$ (see [8]), is also covered by (1.3) with $\mu_\omega = \Sigma_{i\in\mathbb{Z}^n\cap U} q_i(\omega)\delta_{\eta_i(\omega)}$. It is not difficult to see that we can also consider other models like, e.g., the Poisson model (see [17] for more examples).

The models (1.4) and (1.5) correspond to discrete measures μ_ω and results for them about localization, spectral properties or decays can be found in [5, 8, 15, 17, 20]. For Schrödinger equations defined in a lattice, that is $x \in \mathbb{Z}^n$, we refer the reader to [4, 6]. Considering a random time-dependent potential for (1.1), the authors of [3] studied asymptotic behavior of solutions by showing convergence for stochastic Gaussian limits when the two-point correlation function of the potential is rapidly decaying. Still for time random potentials, scaling limits for parabolic waves in random media were investigated in [12]. Despite important progress in the last years, there is still a lack of result for random equations, including Schrödinger ones (see [7]), mainly with respect to the continuum case which seems to be harder-to-treating. Another type of random equations are the parabolic ones, for which we refer the works [9, 10] and their references.

In this paper we find conditions on the potential V_ω for the nonlinear equation (1.2) having solutions almost sure ω. The solution are understood in an integral sense coming from Green functions. From Theorem 3.5 we see how the expected value of the L^∞-norm of solutions depends on the size of potential. Our results also cover continuum random potential like, among others, the examples given in Remark 3.3 and Theorem 3.4. Moreover, we study probabilistic properties like central limit theorem and law of larger numbers for the obtained solutions by independent ensembles. It is worthwhile to mention that, when dealing with the random variable $\omega \mapsto u(x,\omega)$ which maps an element of Ω in the solution of (1.2) associated with the random potential V_ω, we need to extend some known concepts of real random variables for that taking values in a more general Banach space. We refer to Section 2 for more details.

As a further comment, we observe that the random potentials considered in this paper are built from a very general probability space. In this setting does not always make sense to ask what is the probability that the problem (1.2) has an unique solution in $L^\infty(U)$. In order to give some sense to this question we should restrict ourself to probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and random potentials V where the set

$$\{\omega \in \Omega : \text{ the problem (1.2) has a unique solution in } L^\infty(U)\}$$

is an event (measurable). Working in such probability spaces Theorem 3.2 give us immediately a lower bound for the probability that the non-linear problem (1.2) has a unique solution.

The manuscript is organized as follows. In the next section, we introduce some notations, basic definitions and give some properties for an integral operator associated with the random potential V_ω. The results are stated and proved in Section 3.

2 Preliminaries and notation

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a given complete probability space. If $(E, \mathcal{E})$ is a measurable space, any $(\mathcal{F}, \mathcal{E})$-measurable function $X : \Omega \to E$ will be called a E-valued random variable. We use the abbreviation *a.s.* for *almost surely* or *almost sure*.

Let $U \subset \mathbb{R}^n$ be a bounded domain. We adopt the standard notation $\mathcal{M}(U)$ to denote the set of all Random measures over U having finite variation and we call $\mathscr{B}(\mathcal{M}(U))$ the σ-algebra of the borelians of $\mathcal{M}(U)$ generated by the total variation norm. The space of all bounded continuous real-valued functions defined on U will be denoted by $BC(U)$. Since $BC(U)$ is a metric space with the supremum norm, when we refer to a $BC(U)$-valued random variable, the σ-algebra we are considering is always the one generated by the borelians. Similarly to a $\mathcal{X}$-valued Borel random variable $X : \Omega \to \mathcal{X}$, where $\mathcal{X}$ is an arbitrary metric space.

The random potentials considered here are the $BC(U)$-valued random variables defined as follows. Take any random variable $X : \Omega \to \mathcal{M}(U)$ (which is simply a random measure in $\mathcal{M}(U)$) and a fixed function $f \in BC(\mathbb{R}^n)$. Then, for $\mu_\omega = X(\omega)$, the function $V : \Omega \to BC(U)$ defined by

$$V_\omega(x) := \int_U f(x-y)\, d\mu_\omega(y), \quad x \in U,$$

is a $BC(U)$-valued random variable that will be called a random potential. To see that V is a well-defined $BC(U)$-valued random variable, is enough to consider the mapping $T_f : \mathcal{M}(U) \to BC(U)$ given by

$$T_f(\mu)(x) = \int_U f(x-y)\, d\mu(y), \quad x \in U,$$

and to observe that $V = T_f \circ X$. In fact, if we denote by $|\mu|$ the total variation of the measure μ, the inequality

$$\|T_f(\mu)\|_\infty := \sup_{x\in U} |T_f(\mu)(x)| \le \left(\sup_{x\in\mathbb{R}^n} |f(x)| \right) |\mu| \tag{2.1}$$

implies that T_f is a continuous and Borel measurable function. Since V is a composition of two Borel measurable functions, V is a $BC(U)$-valued random variable.

As usual, if $(U, \mathscr{B}, \mu)$ is a measure space, we define

$$\|f\|_{L^\infty(U,d\mu)} = \inf \{a \ge 0 : \mu(\{x : |f(x)| > a\}) = 0\}$$

and the space $L^\infty(U, \mathscr{B}(U), \mu)$ as being the set

$$\{f : U \to \mathbb{R} : f \text{ is Borel measurable and } \|f\|_{L^\infty(U,d\mu)} < \infty\}.$$

When $d\mu = dx$ is the Lebesgue measure in $U \subset \mathbb{R}^n$, we simply denote $L^\infty(U) = L^\infty(U, \mathscr{B}(U), dx)$. Although we are assuming that $f \in BC(\mathbb{R}^n)$, most of the results presented here are also valid if we suppose only the weaker condition $f \in \cap_{\mu \in \mathcal{M}(U-U)} L^\infty(U-U, \mathscr{B}(U-U), \mu)$.

In order to state some convergence results obtained in this paper we need to use the notion of Bochner integrals. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $X : \Omega \to \mathcal{X}$ is a $\mathcal{X}$-valued Borel random variable such that $X = Y$ a.s. in Ω, where $Y : \Omega \to \mathcal{X}$ is a $\mathcal{X}$-valued Borel random variable with $Y(\Omega) \subset \mathcal{X}$ separable, and

$$\int_\Omega \|X(\omega)\|_{\mathcal{X}} \, d\mathbb{P}(\omega) < \infty,$$

then there exist a unique element $\mathbb{E}[X] \in \mathcal{X}$ with the property

$$\ell(\mathbb{E}[X]) = \int_\Omega \ell(X(\omega)) \, d\mathbb{P}(\omega)$$

for all $\ell \in \mathcal{X}^*$, where $\mathcal{X}^*$ is the dual of $\mathcal{X}$. Following the standard notation we write

$$\mathbb{E}[X] = \int_\Omega X(\omega) \, d\mathbb{P}(\omega).$$

We call $\mathbb{E}[X]$ the Bochner integral of X with respect to $\mathbb{P}$. More details about the existence and some properties of this integral can be found in [16, 18]. For these $\mathcal{X}$-valued random variables we define the convergence in probability similarly to the real-valued case, that is, if $\{X_j\}$ is a sequence of $\mathcal{X}$-valued random variable we say that X_j converges to a $\mathcal{X}$-valued random variable X in probability if for all $\varepsilon > 0$, we have

$$\lim_{j\to\infty} \mathbb{P}(\{\omega \in \Omega : \|X_j(\omega) - X(\omega)\|_{\mathcal{X}} \geq \varepsilon\}) = 0. \tag{2.2}$$

When X is real-valued random variable, we use the usual notation and denote the expected value of X and its variance by

$$\mathbb{E}[X] := \int_\Omega X(\omega) \, d\mathbb{P}(\omega) \quad \text{and} \quad \operatorname{Var} X := \mathbb{E}[(\mathbb{E}[X] - X)^2],$$

respectively. For the both senses of expectation presented above we also use the notation

$$\mathbb{E}_A[X] = \int_A X(\omega) \, d\mathbb{P}(\omega),$$

whenever $A \subset \Omega$ is measurable and the right-hand-side of the expression makes sense.

Let X and Y be two E-valued random variable in the same probability space. We say that they are identically distributed if for all $A \in \mathcal{E}$ we have $\mathbb{P}(X^{-1}(A)) = \mathbb{P}(Y^{-1}(A))$. Now we introduce the notion of independence. Given a finite set of random variables $X_1, \ldots X_j$ we say they are independent if for all $A_i \in \mathcal{E}, 1 \leq i \leq j$, we have

$$\mathbb{P}(\cap_{i=1}^j X_i \in A_i) = \prod_{i=1}^j \mathbb{P}(X_i \in A_i).$$

Finally a sequence of random variables $\{X_1, X_2 \ldots\}$ is said independent if all finite collection of this sequence form a set of independent random variables. If $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables we say that $X_1, X_2, \ldots$ are i.i.d. random variables.

3 Main results and proofs

Let G be the Green function of the laplacian operator $-\Delta$ in the bounded domain $U \subset \mathbb{R}^n$ with $n \geq 3$. It is known that, for all x, $y \in U$, there holds

$$0 \leq G(x,y) \leq \frac{1}{n\alpha_n(n-2)} \frac{1}{|x-y|^{n-2}},$$

where α_n stands for the volume of the unit ball in $\mathbb{R}^n$. Hence, if we denote by d_U the the diameter of U, namely

$$d_U := \sup_{x_1,\, x_2 \in U} |x_1 - x_2|,$$

and $B_{d_U}(x) = \{x \in \mathbb{R}^n; |x| < d_U\}$, a straightforward calculation provides

$$\begin{aligned} \int_U G(x,y)dy &\leq \frac{1}{n\alpha_n(n-2)} \int_{B_{d_U}(x)} \frac{1}{|x-y|^{n-2}} dy \\ &= \frac{1}{n\alpha_n(n-2)} \frac{n\alpha_n d_U^2}{2} = \frac{d_U^2}{2(n-2)}, \end{aligned} \tag{3.1}$$

for all $x \in U$. From now on we write only $l_0 = l_0(n, U)$ to denote the following quantity

$$l_0 := \frac{d_U^2}{2(n-2)}. \tag{3.2}$$

Inequality (3.1) implies that is well defined the map $H : L^\infty(U) \to L^\infty(U)$ given by

$$H(\varphi)(x) := \int_U G(x,y)\varphi(y)dy, \quad x \in U.$$

More specifically, for any $\varphi \in L^\infty(U)$, there holds

$$|H(\varphi)(x)| \leq \int_U G(x,y)|\varphi(y)|dy \leq \|\varphi\|_\infty \int_U G(x,y)dy$$

and therefore

$$\|H(\varphi)\|_\infty \leq l_0 \|\varphi\|_\infty. \tag{3.3}$$

Standard calculations show that the problem (1.2) is formally equivalent to the integral equation

$$u(x) = H(g) + H(V_\omega u) + H(bu|u|^{p-1}). \tag{3.4}$$

In what follows we make suitable estimates on the terms of the integral equation in order to be able to apply a fixed point argument. We first set $\mathcal{X} := L^\infty(U)$ and define, for any fixed $\omega \in \Omega$, the linear function $T : \mathcal{X} \to \mathcal{X}$ by

$$T(u) := H(V_\omega u), \quad \forall\, u \in \mathcal{X}.$$

It follows from (3.3) and (2.1) that, for any $u \in \mathcal{X}$, there holds

$$\|T(u)\|_\infty \le l_0 \|V_\omega u\|_\infty \le l_0 \|f\|_\infty \|\mu_\omega\| \, \|u\|_\infty, \tag{3.5}$$

and therefore

$$\|T\|_\infty \le l_0 \|f\|_\infty \|\mu_\omega\|.$$

For the nonlinear term we define $B : \mathcal{X} \to \mathcal{X}$ by setting

$$B(u) := H(b|u|^{p-1}u), \quad \forall\, u \in \mathcal{X}.$$

If a_1, $a_2 \in \mathbb{R}$ there holds

$$\left| a_1|a_1|^{p-1} - a_2|a_2|^{p-1} \right| \le p|a_1 - a_2| \left(|a_1|^{p-1} - |a_2|^{p-1} \right),$$

and therefore it follows that

$$\|b(\cdot)\left(u|u|^{p-1} - \tilde{u}|\tilde{u}|^{p-1} \right)\|_\infty \le \|b\|_\infty \|u - \tilde{u}\|_\infty \left(\|u\|_\infty^{p-1} - \|\tilde{u}\|_\infty^{p-1} \right).$$

This inequality and the same argument used in (3.5) imply that

$$\|B(u) - B(\tilde{u})\|_\infty \le l_0 p \|b\|_\infty \|u - \tilde{u}\|_\infty \left(\|u\|_\infty^{p-1} - \|\tilde{u}\|_\infty^{p-1} \right), \tag{3.6}$$

for any u, $\tilde{u} \in L^\infty(U)$.

All together, the above estimates enable us to solve the random equation (1.2) as follows.

Proposition 3.1. *Given f, $b, g \in L^\infty(U)$ and $\omega \in \Omega$, we consider the potential V_ω induced by the random measure $\mu_\omega := X(\omega)$. Let l_0 be the quantity introduced in* (3.2) *and set*

$$\tau_\omega := l_0 \|f\|_\infty \|\mu_\omega\| \quad \textit{and} \quad K := l_0 p \|b\|_\infty. \tag{3.7}$$

If $\varepsilon > 0$ and $\omega \in \Omega$ are such that

$$0 \le \tau_\omega < 1, \qquad \frac{2^p K \varepsilon^{p-1}}{(1 - \tau_\omega)^{p-1}} + \tau_\omega < 1, \tag{3.8}$$

and $\|g\|_\infty \le \varepsilon / l_0$, then the equation (1.2) *has a unique integral solution*

$$u_\omega = u(\cdot, \omega) \in L^\infty(U) \ \textit{such that} \ \|u_\omega\|_\infty \le \frac{2\varepsilon}{1 - \tau_\omega}. \tag{3.9}$$

Proof. For each $\omega \in \Omega$, we consider the closed ball

$$\mathcal{B}_\varepsilon = \left\{ u \in L^\infty(U); \|u\|_\infty \le \frac{2\varepsilon}{(1 - \tau_\omega)} \right\}$$

endowed with the metric $d(u, v) := \|u - v\|_\infty$. We are going to show that the map

$$\Phi(u) := H(g) + H(V_\omega u) + H(bu\,|u|^{p-1}) = H(g) + T(u) + B(u) \tag{3.10}$$

is a contraction on the complete metric space $(\mathcal{B}_\varepsilon, d)$. Using the estimates (3.3), (3.5), and (3.6) with $\tilde{u} = 0$, we obtain

$$\begin{aligned}\|\Phi(u)\|_\infty &\leq \|H(g)\|_\infty + \|T(u)\|_\infty + \|B(u)\|_\infty \\ &\leq l_0 \|g\|_\infty + \tau_\omega \|u\|_\infty + K\|u\|_\infty^p \\ &\leq \varepsilon + \tau_\omega \frac{2\varepsilon}{1-\tau_\omega} + \frac{2^p K \varepsilon^p}{(1-\tau_\omega)^p} \\ &= \left(1 + \tau_\omega + \frac{2^p K \varepsilon^{p-1}}{(1-\tau_\omega)^{p-1}}\right) \frac{\varepsilon}{1-\tau_\omega}\end{aligned}$$

for all $u \in \mathcal{B}_\varepsilon$ and $\omega \in \Omega$. Hence, it follows from (3.8) that

$$\|\Phi(u)\|_\infty \leq \frac{2\varepsilon}{1-\tau_\omega}.$$

This shows that Φ maps $\mathcal{B}_\varepsilon$ into $\mathcal{B}_\varepsilon$.

For any $u, \widetilde{u} \in \mathcal{B}_\varepsilon$, it follows from (3.5) and (3.6) that

$$\begin{aligned}\|\Phi(u) - \Phi(\widetilde{u})\|_\infty &= \|T(u-\widetilde{u})\|_\infty + \|B(u) - B(\widetilde{u})\|_\infty \\ &\leq \tau_\omega \|u - \widetilde{u}\|_\infty + K\|u-\widetilde{u}\|_\infty \left(\|u\|_\infty^{p-1} + \|\widetilde{u}\|_\infty^{p-1}\right) \\ &\leq \left(\tau_\omega + \frac{2^p K \varepsilon^{p-1}}{(1-\tau_\omega)^{p-1}}\right) \|u - \widetilde{u}\|_\infty.\end{aligned}$$

Recalling (3.8), the above estimate implies that the map Φ is a contraction. The Banach fixed point theorem assures that there is a unique solution u for the integral equation (3.4) such that $\|u\|_\infty \leq (2\varepsilon)/(1-\tau_\omega)$.

$\blacksquare$

The next results are related to the randomness introduced by the random potential V and the existence and uniqueness of solutions for the problem (1.2). Roughly speaking, we first obtain the probability of (1.2) having a solution given by the method discussed above. In the sequel we study two important limit theorems in probability theory, namely, the central limit theorem and the law of large numbers for a sequence of random potentials.

Theorem 3.2. *Let ν be the probability measure induced on $\mathbb{R}$ by the random variable $\omega \mapsto \| \mu_\omega \|$. Let $g \in L^\infty(U)$ be such that $\|g\|_\infty < \frac{1}{l_0}(\frac{1}{2^p K})^{\frac{1}{p-1}}$, where $K = l_0 p \|b\|_\infty$. Choose $0 < c_0 < 1$and set*

$$\varepsilon_0 := \left(\frac{(1-c_0)^p}{2^p K}\right)^{\frac{1}{p-1}}.$$

Let $\mathcal{A}$ be the set of $\omega \in \Omega$ such that (1.2) has a unique solution $u(\cdot, \omega)$ given by Proposition 3.1 with $\varepsilon = \varepsilon_0$. The set $\mathcal{A}$ is called the admissible one for the random variable X.

(i) The set $\mathcal{A}$ is $\mathcal{F}$-measurable and the probability of (1.2) having a solution is

$$\mathbb{P}(\mathcal{A}) = \nu\left(\left[0, \frac{1}{l_0\|f\|_\infty}\right)\right).$$

(ii) Let u_ω, $\tilde{u}_\omega$ be two solutions of (1.2) *corresponding, respectively, to $\mu_\omega, g, \mathcal{A}$ and $\tilde{\mu}_\omega, \tilde{g}, \widetilde{\mathcal{A}}$. Assume that $\mathcal{A} \cap \widetilde{\mathcal{A}} \neq \varnothing$ and define, for $\omega \in \mathcal{A} \cap \widetilde{\mathcal{A}}$,*

$$\eta_\omega := l_0 \|f\|_\infty \max\{|\mu_\omega|, |\widetilde{\mu}_\omega|\}.$$

We have that

$$\|u(\cdot,\omega) - \tilde{u}(\cdot,\omega)\|_\infty \leq \frac{l_0 \left(\|g - \tilde{g}\|_\infty + \dfrac{2\varepsilon_0}{1-\eta_\omega} \|f\|_\infty |\mu_\omega - \tilde{\mu}_\omega| \right)}{1 - \eta_\omega - \dfrac{2^p K \varepsilon_0^{p-1}}{(1-\eta_\omega)^{p-1}}} \tag{3.11}$$

for all $\omega \in \mathcal{A} \cap \widetilde{\mathcal{A}}$.

(iii) The map $\mathcal{U} : \mathcal{A} \to L^\infty(U)$ given by $\mathcal{U}(\omega) := u(\cdot, \omega)$ is a random variable and there holds

$$\|u(\cdot,\omega)\|_\infty \leq \frac{2\varepsilon_0}{1 - \tau_\omega} = 2\varepsilon_0 \sum_{j=0}^{\infty} \tau_\omega^j, \tag{3.12}$$

for all $\omega \in \mathcal{A}$.

Proof. We first notice that the choice of ε_0 implies that $\|g\|_\infty \leq \varepsilon_0 / l_0$. Moreover, $\omega \in \mathcal{A}$ if only if $\tau_\omega = l_0 \|f\|_\infty |\mu_\omega|$ verifies (3.8) with $\varepsilon = \varepsilon_0$. Then, if $Y(\omega) = |X(\omega)| = |\mu_\omega|$, it follows that $\mathcal{A} = \left\{ Y \in \left[0, \frac{1}{l_0\|f\|_\infty}\right) \right\}$ is measurable and

$$\begin{aligned} \mathbb{P}(\mathcal{A}) &= \mathbb{P}\left(Y \in \left[0, \frac{1}{l_0 \|f\|_\infty}\right) \right) = \mathbb{P}_Y\left(\left[0, \frac{1}{l_0 \|f\|_\infty}\right) \right) \\ &= \nu\left(\left[0, \frac{1}{l_0 \|f\|_\infty}\right) \right). \end{aligned}$$

This establishes (i).

Now we deal with item (ii). Firstly, observe that $\eta_\omega = \max\{\tau_\omega, \widetilde{\tau}_\omega\}$, where

$$\tau_\omega = l_0 \|f\|_\infty |\mu_\omega| \text{ and } \widetilde{\tau}_\omega = l_0 \|f\|_\infty |\tilde{\mu}_\omega|.$$

Subtracting the integral equations verified by u_ω and $\tilde{u}_\omega$, and afterwards computing $\|\cdot\|_\infty$, we obtain

$$\begin{aligned} \|u_\omega - \tilde{u}_\omega\|_\infty &\leq \|H(g - \tilde{g})\|_\infty + \|H(V_\omega(u - \tilde{u}_\omega))\|_\infty \\ &\quad + \|H((V_\omega - \widetilde{V}_\omega)\tilde{u}_\omega)\|_\infty \\ &\quad + \left\| H(b\left(u_\omega |u_\omega|^{p-1} - \tilde{u}_\omega |\tilde{u}_\omega|^{p-1}\right)) \right\|_\infty \\ &\leq l_0 \|g - \tilde{g}\|_\infty + l_0 \|f\|_\infty |\mu_\omega| \|u_\omega - \tilde{u}_\omega\|_\infty \\ &\quad + l_0 \|f\|_\infty |\mu_\omega - \tilde{\mu}_\omega| \|\tilde{u}_\omega\|_\infty \\ &\quad + l_0 p \|b\|_\infty \|u_\omega - \tilde{u}_\omega\|_\infty (\|u_\omega\|_\infty^{p-1} - \|\tilde{u}_\omega\|_\infty^{p-1}). \end{aligned}$$

It follows from (3.9) that

$$\|u_\omega\|_\infty \leq \frac{2\varepsilon_0}{1 - \tau_\omega} \leq \frac{2\varepsilon_0}{1 - \eta_\omega} \text{ and } \|\tilde{u}\|_\infty \leq \frac{2\varepsilon_0}{1 - \widetilde{\tau}_\omega} \leq \frac{2\varepsilon_0}{1 - \eta_\omega}.$$

The two above expressions give us

$$\begin{aligned}
\|u_\omega - \tilde{u}_\omega\|_\infty &\le l_0\|g-\tilde{g}\|_\infty + l_0\|f\|_\infty |\!|\!| \mu_\omega |\!|\!| \, \|u_\omega - \tilde{u}_\omega\|_\infty \\
&\quad + l_0 \frac{2\varepsilon_0}{1-\eta_\omega}\|f\|_\infty |\!|\!| \mu_\omega - \tilde{\mu}_\omega |\!|\!| + \frac{2^p K \varepsilon_0^{p-1}}{(1-\eta_\omega)^{p-1}}\|u_\omega - \tilde{u}_\omega\|_\infty \\
&= l_0\|g-\tilde{g}\|_\infty + l_0 \frac{2\varepsilon_0}{1-\eta_\omega}\|f\|_\infty |\!|\!| \mu_\omega - \tilde{\mu}_\omega |\!|\!| \\
&\quad + \left[\eta_\omega + \frac{2^p K \varepsilon_0^{p-1}}{(1-\eta_\omega)^{p-1}}\right]\|u_\omega - \tilde{u}_\omega\|_\infty ,
\end{aligned}$$

which yields (3.11).

Taking $\mu_\omega, \tilde{\mu}_\omega$ independent of ω, i.e. $\mu_\omega = \mu$ and $\tilde{\mu}_\omega = \tilde{\mu}$, for all $\omega \in \Omega$, we see from (3.7) and (3.11) that the data-map solution $\mathcal{L}(\mu, g) = u$ is continuous from

$$\left\{(\mu, g) \in \mathcal{M}(U) \times L^\infty(U);\ |\!|\!| \mu |\!|\!| < \frac{1}{l_0\|f\|_\infty}, \|g\|_\infty < \frac{1}{l_0}\left(\frac{1}{2^p K}\right)^{\frac{1}{p-1}}\right\} \text{ to } L^\infty(U), \tag{3.13}$$

where u is the deterministic solution of (1.2) corresponding to the data (μ, g). From this, and because $X|_{\mathcal{A}}$ given by $X(\omega) = \mu_\omega$ is measurable, it follows that the composition $\mathcal{U}(\omega) = \mathcal{L}(\mu_\omega, g) = \mathcal{L}(X(\omega), g)$ from $\mathcal{A}$ to $L^\infty(U)$ is measurable.

In view of the series $\frac{1}{1-z} = \sum_{j=0}^\infty z^j$ for $|z| < 1$, we finish by observing that (3.12) follows at once from (3.9) with $\varepsilon = \varepsilon_0$ and $\omega \in \mathcal{A}$.

■

Remark 3.3. *Here we give examples of random potentials for which there exists solution almost surely in Ω. The first setting occurs if we suppose that the measure ν has compact support contained in the interval $[0, a]$, with $a < \frac{1}{l_0\|f\|_\infty}$. In this case it follows from the first item of the above theorem that $\mathbb{P}(\mathcal{A}) = 1$, i.e., the solution exists almost surely in Ω. Secondly, we take $\{\mu_j\}_{j\in\mathbb{N}}$ a sequence in $\mathcal{M}(U)$ and let $\{a_j(\omega)\}_{j\in\mathbb{N}}$ be a sequence of random variables from Ω to $\mathbb{R}$. Consider the random variable μ_ω defined by*

$$\mu_\omega = \sum_{j=1}^\infty a_j(\omega)\mu_j.$$

For $q > 1$, suppose that

$$|a_j(\omega)| < \frac{(\sum_{k=1}^\infty \frac{1}{k^q})^{-1}}{l_0 |\!|\!| \mu_j |\!|\!| \, \|f\|_\infty} \cdot \frac{1}{j^q} \ \ a.s.\ in\ \Omega,$$

for all $j \in \mathbb{N}$. Then

$$|\!|\!| \mu_\omega |\!|\!| \le \sum_{j=1}^\infty |a_j(\omega)| |\!|\!| \mu_j |\!|\!| < \frac{1}{l_0\|f\|_\infty} \ \ a.s.\ in\ \Omega,$$

and Theorem 3.2 assures that there is an integral solution for (1.2) a.s. in Ω.

In the sequel we show how the Borel-Cantelli's Lemma can be used to give a sufficient condition for the existence of solution a.s. in Ω.

Theorem 3.4. *Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{M}(U)$ and let $\{a_j(\omega)\}_{j\in\mathbb{N}}$ be a sequence of random variables from Ω to $\mathbb{R}$. Assume that the following series is convergent in $\mathcal{M}(U)$*

$$\mu_\omega = \sum_{j=1}^{\infty} a_j(\omega)\mu_j.$$

For any $k \in \mathbb{N}$ define

$$S_k(\omega) = \sum_{j=1}^{k} a_j(\omega)\mu_j$$

and $L_k = \{\omega \in \Omega : \| S_k \| \geq \tilde{c}\}$, with $0 < \tilde{c} < 1/(l_0\|f\|_\infty)$. If

$$\sum_{k=1}^{\infty} \mathbb{P}(L_k) < \infty$$

then there is an integral solution for (1.2) *almost surely in Ω.*

Proof. By the Borel-Cantelli's Lemma we get that $\mathbb{P}(\limsup L_k) = 0$, that is,

$$\mathbb{P}\left(\cup_{j=1}^{\infty} \cap_{k=j}^{\infty} \{\| S_k \| < \tilde{c}\}\right) = 1$$

It follows that, for almost sure ω, there is $j_0 = j_0(\omega)$ such that for all $j > j_0$, we have

$$\| S_k(\omega) \| < \tilde{c}.$$

Therefore by taking the limit when k goes to infinity, we obtain

$$\| \mu_\omega \| = \lim_{j\to\infty} \| S_j(\omega) \| \leq \tilde{c} < \frac{1}{l_0\|f\|_\infty} \quad \text{a.s. in } \Omega.$$

This inequality and Theorem 3.2 imply that there is an integral solution $u(x, \omega)$ for (1.2) almost surely in Ω.

■

A straightforward calculation shows that in general $\mathbb{E}_\Omega(u(x,\omega))$ does not satisfies the equation (1.2), even if we replace the random potential by its mean. However, we are able to obtain some information on the average and moments of the random solution u_ω previously obtained. It is worthwhile to mention that, when dealing with the random variable $\omega \mapsto u_\omega$, the expectation has to be understood in the Bochner sense (see Section 2). Note also that a solution $u_\omega \in L^\infty(U)$ for (3.4) in fact belongs to the separable subspace $C(\overline{U})$.

Theorem 3.5. *Under hypotheses of Theorem 3.2 let us denote by $u_\omega(x) = u(x,\omega) \in \mathcal{A}$ the solution of* (1.2). *Let $m \in \mathbb{N}$ and suppose that*

$$\sum_{j=1}^{\infty} \frac{(m+j-1)!}{(m-1)!j!} (l_0\|f\|_\infty)^j \ \mathbb{E}_{\mathcal{A}}[\, \| \mu_\omega \|^j \,] < +\infty. \tag{3.14}$$

Then $\mathbb{E}_{\mathcal{A}}[|u|^m(x,\omega)] \in L^\infty(U)$ *and*

$$\mathbb{E}_{\mathcal{A}}\left[\||u|^m(\cdot,\omega)\|_{L^\infty(U)}\right] < \infty. \tag{3.15}$$

In particular, $\mathbb{E}_{\mathcal{A}}[u(x,\omega)] \in L^\infty(U)$.

Proof. It follows from (3.12) that

$$\||u|^m(\cdot,\omega)\|_{L^\infty(U)} \leq \|u(\cdot,\omega)\|^m_{L^\infty(U)} \leq \frac{(2\varepsilon_0)^m}{(1-\tau_\omega)^m}. \tag{3.16}$$

Recalling that $\tau_\omega = l_0\|f\|_\infty \lvert\mu_\omega\rvert$ and computing $\mathbb{E}_{\mathcal{A}}$ in (3.16), we obtain

$$\begin{aligned}\|\mathbb{E}_{\mathcal{A}}\left[|u|^m(x,\omega)\right]\|_{L^\infty(U)} &\leq \mathbb{E}_{\mathcal{A}}\left[\||u|^m(x,\omega)\|_{L^\infty(U)}\right] \\ &\leq (2\varepsilon_0)^m \mathbb{E}_{\mathcal{A}}\left[\left(1+\sum_{j=1}^{\infty}\frac{(m+j-1)!}{(m-1)!j!}\tau_\omega^j\right)\right]\end{aligned}$$

By using the linearity of the expectation and definition of τ_ω we get the following upper bound for the right hand side above

$$(2\varepsilon_0)^m + (2\varepsilon_0)^m \sum_{j=1}^{\infty}\frac{(m+j-1)!}{(m-1)!j!}\left(l_0\|f\|_\infty\right)^j \mathbb{E}_{\mathcal{A}}\left[\, \lvert\mu_\omega\rvert^j \,\right],$$

which is finite due to (3.14). The last assertion of the statement follows from (3.15) with $m=1$ and the easy estimate

$$\|\mathbb{E}_{\mathcal{A}}\left[u(x,\omega)\right]\|_{L^\infty(U)} \leq \mathbb{E}_{\mathcal{A}}\left[\||u|(x,\omega)\|_{L^\infty(U)}\right].$$

■

3.1 Classical Probability Limit Theorems

We start this section by recalling basic background concerning to some main limit theorems in probability. A real-valued random variable $X:\Omega\to\mathbb{R}$ in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ has standard normal distribution, notation $X\sim N(0,1)$, if for all $x\in\mathbb{R}$ its cumulative distribution function verifies

$$\mathbb{P}(X\leq x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt.$$

A sequence of real-valued random variable $\{Y_j\}_{j\in\mathbb{N}}$ in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is said to converge in distribution to a standard normal random variable, notation $Y_j \to N(0,1)$, if for all $x\in\mathbb{R}$ we have

$$\lim_{j\to\infty}\mathbb{P}(Y_j\leq x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt.$$

In the sequel we show versions of the central limit theorem and a weak law of large numbers for the random $L^\infty(U)$-solutions obtained in Section 2.

Theorem 3.6. *Let $\{X_j\}_{j\in\mathbb{N}}$ be an independent identically distributed (i.i.d.) sequence of random variables $X_j : \Omega \to \mathcal{M}(U)$. Assume that the admissible set $\mathcal{A}_j = \Omega$ for all j, and let $u_j(\cdot,\omega) \in L^\infty(U)$ be the solution given by Theorem 3.2 with respect to $X_j(\omega) = \mu_{\omega,j}$ and g. We have that $\{Z_j\}_{j\in\mathbb{N}}$ given by $Z_j(\omega) := \|u_j(\cdot,\omega)\|_\infty$ is a i.i.d. sequence of random variables, and if $m = \mathbb{E}[\|u_j(\cdot,\omega)\|_\infty] < \infty$ and $\sigma^2 := Var Z_j < \infty$ then following holds as $k \to +\infty$*

$$\sum_{j=1}^{k} \frac{(Z_j - m)}{\sigma\sqrt{k}} \to N(0,1).$$

Proof. Recall the data-solution map $\mathcal{L}(\mu, g)$ defined in the proof of Theorem 3.2 (see (3.13)). Fixed g such that $\|g\|_\infty < \frac{1}{l_0}(\frac{1}{2^pK})^{\frac{1}{p-1}}$, consider

$$S_g(\mu) = \mathcal{L}(\mu, g) \tag{3.17}$$

defined from D to $L^\infty(U)$, where $D = \left\{\mu \in \mathcal{M}(U) : |\mu| < \frac{1}{l_0\|f\|_\infty}\right\}$. Since $\|\cdot\|_\infty$ is continuous from $L^\infty(U)$ to $\mathbb{R}$ and

$$Z_j(\omega) = \|u_j(\cdot,\omega)\|_\infty = \|S_g \circ X_j(\omega)\|_\infty,$$

we get that $\{Z_j\}_{j\in\mathbb{N}}$ is a i.i.d. sequence. The convergence stated in the theorem follows from the central limit theorem.

∎

Theorem 3.7. *Let $\{X_j\}_{j\in\mathbb{N}}$ be an independent sequence of random variables $X_j : \Omega \to \mathcal{M}(U)$. Assume that the admissible set $\mathcal{A}_j = \Omega$ for all j, and let $u_j(\cdot,\omega) \in L^\infty(U)$ be the solution given by Theorem 3.2 with respect to $X_j(\omega) = \mu_{\omega,j}$ and g. If $X_j \to X$ a.s. and*

$$L = \sup_{j\in\mathbb{N}} \left(\operatorname*{ess\,sup}_{\omega\in\Omega} |\mu_{\omega,j}| \right) < \frac{1}{l_0\|f\|_\infty}, \tag{3.18}$$

then

$$\sum_{j=1}^{k} \frac{u_j(x,\omega) - \mathbb{E}_\Omega[u_j(x,\omega)]}{k} \to 0 \tag{3.19}$$

and

$$\sum_{j=1}^{k} \frac{\|u_j(\cdot,\omega)\|_\infty - \mathbb{E}_\Omega[\|u_j(\cdot,\omega)\|_\infty]}{k} \to 0, \tag{3.20}$$

when $k \to \infty$, where the convergence in (3.19) and (3.20) are in probability sense (see (2.2)).

Proof. Notice that $X_j \to X$ a.s. is equivalent to $\mu_{\omega,j} \to \mu_\omega = X(\omega)$ in $\mathcal{M}(U)$ almost surely. From this and the continuity of data-solution map $\mathcal{L}(\cdot,\cdot)$ (see (3.13)), it follows that

$$\|u_j(\cdot,\omega) - u(\cdot,\omega)\|_\infty = \|\mathcal{L}(\mu_{\omega,j}, g) - \mathcal{L}(\mu, g)\|_\infty \to 0,$$

when $j \to \infty$. Recalling (3.12) and afterwards using (3.18), we obtain

$$\begin{aligned} \|u_j(\cdot,\omega)\|_\infty &\le \frac{2\varepsilon_0}{1 - l_0\|f\|_\infty(\operatorname{ess\,sup}_{\omega\in\Omega}\|\mu_{\omega,j}\|)} \\ &\le \frac{2\varepsilon_0}{1-L} = Q_0, \text{ a.s. in } \Omega. \end{aligned} \tag{3.21}$$

Since X_j's are independent, it follows that $\{Y_j\}_{j\in\mathbb{N}}$ defined by $Y_j = \|u_j(\cdot,\omega)\|_\infty = \|S_g \circ X_j(\omega)\|_\infty$ are also independent, where S_g is as in (3.17). So, from Chebyshev's inequality and the independence of $\{Y_j\}_{j\in\mathbb{N}}$, we have that

$$\begin{aligned} &\mathbb{P}\left(\left|k^{-1}\sum_{j=1}^{k}(\|u_j(\cdot,\omega)\|_\infty - \mathbb{E}_\Omega[\|u_j(\cdot,\omega)\|_\infty])\right| \ge \delta\right) \\ &\quad\le \frac{1}{(k\delta)^2}\mathbb{E}_\Omega\left[\left|\sum_{j=1}^{k}(\|u_j(\cdot,\omega)\|_\infty - \mathbb{E}_\Omega[\|u_j(\cdot,\omega)\|_\infty]\,)\right|^2\right] \\ &\quad= \frac{1}{(k\delta)^2}\sum_{j=1}^{k}\mathbb{E}_\Omega\left[|(\|u_j(\cdot,\omega)\|_\infty - \mathbb{E}_\Omega[\|u_j(\cdot,\omega)\|_\infty]\,)|^2\right] \\ &\quad\le \frac{1}{(k\delta)^2}\sum_{j=1}^{k}\mathbb{E}_\Omega\left[|2Q_0|^2\right] \le \frac{4Q_0^2}{\delta^2}\frac{1}{k}, \end{aligned}$$

where we have used (3.21). Letting $k \to +\infty$ in the above expression we get (3.20). The convergence (3.19) can be proved with similar arguments.

■

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